Swami Ramanand Teerth Marathwada University,
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STUDY OF FRACTIONAL DIFFUSION – WAVE EQUATIONS AND ITS APPLICATIONS

Research Proposal Submitted

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1 Introduction to proposed research topic

Fractional Calculus is becoming a useful and key tool in the analysis of scientific problems in a broad array of fields such as physics, engineering, biology and economics. In particular, fractional partial differential equations have turned out to be especially relevant. For example, fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations [1].

The fractional diffusion equations has been explicitly introduced to describe diffusion in media whereas fractional wave equations governs the propagation of mechanical diffusive waves in viscoelastic media. Fractional diffusion-wave equation is a generalization of fractional diffusion equations and fractional wave equations. Fractional diffusion-wave equation have been successfully used to describe diffusion process and have been proposed to deal with viscoelastic problems [8].

For example, in our research we will study the following type of diffusion-wave equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = D\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}},\ 0 \le x \le L,\ 0 < \alpha \le 2, 1 < \beta \le 2$$

$$initial\ condition : u(x,0) = \phi_0(x), 0 \le x \le L$$

$$boundary\ conditions : u(0,t) = \phi_1(t),\ \ u(L,t) = \phi_2(t), t > 0$$

where u = u (x, t) is the potential (e.g. temperature, velocity, concentration) defined in space and time, x and t are the space and time variables, α and β are the parameters describing fractional derivative order, and D is a coefficient.

If $0 < \alpha \le 1$, $1 < \beta \le 2$, it represents fractional order diffusion equations.

If $1 < \alpha \le 2$, $1 < \beta \le 2$, it represents fractional order wave equations.

In order for this fractional formalism to be useful in solving practical problems one should be able to know how to find either exact or at least approximate solutions of these fractional equations. Therefore, the proposed and study of efficient, accurate and easy to implement numerical and analytical methods are quite important.

2 Purpose of carrying out proposed research work:

The purpose of the research is to explore the behavior of the fractional diffusion equations, fractional wave equations, diffusion-wave equations and equations of

these kinds, from the perspective of fractional calculus. Fractional calculus, integration and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concepts of rate of change and area under the curve. This provides many useful applications in science and technology. We will develop higher order numerical methods for fractional diffusion equations, fractional wave equations and diffusion-wave equation. Also, we will study the convergence and stability of these methods. Further, solutions of the test problems will be simulated by Mathematica software.

3 Survey or review of literature and development in the relevant subject:

The modelling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [6]. An illustration of this are the generalization of the fractional partial difference equation suggested as a replacement of Fick's law [7], the fractional-order diffusion equation studied by Metzler, Glockle and Nonnenmacher [8].

A space-time fractional diffusion equation obtained from the standard diffusion equation by replacing the second order space derivative by a fractional Riesz derivative, and the first order time-derivative by a Caputo fractional derivative, has been treated by Saichev and Zaslavsky, Uchajkin and Zolotarev, Gorenflo and Mainardi, Metzler and Klafter. The idea of fractional differentiation is attributed first to a series of letters written between Leibnitz and L'Hospital [1, 2, 3].

Numerical methods and analysis of the fractional order partial differential equations are limited to date. Adomian decomposition method (ADM) and variational iteration method (VIM) are widely used iterative methods for obtaining solutions in differential equations. Adomian's decomposition method is of great interest to applied sciences today. George Adomian proposed the Decomposition method in the 1980's to solve nonlinear function equations. This method can widely handle quite general nonlinearities. The variational iteration method (VIM) was first proposed by Chinese researcher Ji-Haun He (2006) and further developed by him (He, 2006, 2008, 2012; He and Wu, 2007) used to obtain an approximate analytical solutions for nonlinear problems [4].

4 Research gap(s)

Fractional calculus provides a natural interpolation between different equations. For instance, we have the fractional interpolation between the wave equation (hyperbolic) and the diffusion equation (parabolic). Thus, we have the family of fractional equations

$$D_t^{\alpha} u(x,t) - \lambda \; D_{xx}^2 u(x,t) = 0, \; 0 < x < L, \; t > 0 \qquad (4.1)$$

when $\alpha = 2$, we have the wave equation, while we have the diffusion equation for $\alpha = 1$. The study of such possible interpolations has not been fully exploited yet for $0 < \alpha < 1$.

The different equations above can be fractionalized by replacing the integer derivatives either in space or time by fractional derivatives. The range of fractional equations is very wide at the same time that the exploration of the associated dynamics and solutions is a task to be fulfilled. On the other hand, it is an open issue to analyze the relation of the different fractionalizations at the discrete and the continuous levels.

5 Importance of Study:

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade Fractional calculus has been applied to almost every field of science, engineering and mathematics [5.9.10].

International status: Many authors like Leibnitz (1695), Euler (1730), Lagrange (1772), Laplace (1812), Fourier (1822), Abel (1823), Liouvielli (1832), Grunwald (1867), Riemann (1876), Riesz (1949), K.B.Oldham and J.Spanier (1974), Miller and Ross (1993), Igor Podlubny (1999) and many more contributed in the development of fractional calculus and fractional differential equations. Several finite difference methods (FDM) have been developed for differential equations including FDM by Richtmeyer and Morton (1967), Smith (1978), Anderson, Tannehill and Pletcher (1984), Berezin and Zhidkov (1965) and Roache (1976).

National status: Researchers from India also contributed remarkably in the development of fractional calculus and its applications in various fields. To name

a few, R.K.Sexana (2008), V.Srivastava and K.N.Rai (2009), S. Das, K.Gupta and S.Saha Ray (2011), V.B.L. Chaurasia and Devendera Kumar (2011) and Pradeep Kumar, Dwijendra N Pandey, D.Bahuguna (2014) developed various analytical and numerical method to study different fractional differential equations. Hradyesh Kumar Mishra and Atulya K. Nagar (2011) developed He-Laplace method for linear and Nonlinear Partial differential equation. Modified ADM have been investigated by Hardik S. Patel and Rama Kanta Meher (2015) for initial boundary value problems and further Randir Singh (2017) studied ADM for solving derivative-dependent doubly singular boundary value problem.

Regional status: Many scientists and researchers from various institutes of Maharashtra have been actively involved in the study of fractional calculus and contributed a lot. Kiran M. Kolwankar (1997), A.D.Gangal (1998), A.K.Pani and PSV Nataraj (2010) studied fractional differential equations and models involving fractional differential equations. Various analytical and numerical methods to solve fractional differential equations have been intensively studied by S.Bhalekar and V.Daftardar-Gejji (2011). K.C.Takale (2012), J.J.Tripathi, S.D.Warbhe, K.C.Deshmukh and J.Verma (2017) prepared various mathematical models for fractional differential equations.

Scope of Study: The analysis of wave and diffusion equation field is a task central to many applications. Wave fields are encountered in acoustic, radar, and geophysics to name a few. Diffusion fields are found, for example, in physics, chemistry and biology. For this purpose, we will study partial differential equations and fractional partial differential equations. We will study wave equations, diffusion equations and also, numerical techniques to find their solutions. We will study the powerful mathematical software Mathematica to simulate the solutions graphically. Further we will study fractional wave equations, fractional diffusion equations and fractional diffusion-wave equations. We will develop iterative and finite difference schemes to solve them. Solutions of these equations will be simulated graphically. We intend to publish the work in reputed Journals.

6 Objectives of proposed research work

(i) To study linear and nonlinear partial differential equations.

(ii) To study linear and nonlinear fractional partial differential equations.

(iii) To study fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations.

(iv) To study analytical and numerical methods for solving linear and nonlinear fractional partial differential equations.

(v) To obtain analytical solutions of fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations, if possible. Furthermore, we obtain the approximate solutions of fractional partial differential equations by developed fractional order finite difference schemes and compare these solutions with their analytical solution, if it is possible.

(vi) To discuss convergence and stability of solutions obtained by developed numerical methods.

(vii) To simulate the solutions of fractional partial differential equations using mathematical software Mathematica and to discuss the nature of the solutions.

(viii) To study the recent applications of fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations in various fields of science and technology.

(ix) To publish research papers in reputed journals.

7 Hypothesis:

Numerical and iterative methods are efficient tools to study fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations.

8 Limitation(s) of the work:

In our research, we consider the fractional differential equations. The fractional order model are more adequate than integer-order model, because fractional derivatives and integrals enable the description of memory properties of various materials and processes. The analytical solution of most fractional differential equations cannot be obtained, and as a consequence, approximate and numerical techniques are playing important role in identifying the solution behavior of such fractional equation.

In our research, approximate and numerical techniques mainly includes ADM, VIM and FDM to find solutions of the test problems. The effectiveness of the computational methods is dependent on the possibility of determining the Euler's continuous gamma functions and depends on the possibility of calculating Mittag-Leffler functions $E_\alpha(z)$, and satisfying CFL conditions in finite difference methods. The function $E_\alpha(z)$ was first introduced in 1903 by Mittag-Liffler.

9 Research Methodology:

Research Methodology will include the following steps:

(i) To refer the relevant reference books on partial differential equations and fractional partial differential equations.

(ii) To study Mathematica software.

(iii) To obtain analytical solutions of partial differential equations using Mathematica software and study its various applications.

(iv) To obtain the approximate solutions of diffusion equation and wave equation by different numerical methods and represent their solutions graphically by Mathematica software.

(v) To study fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations. Further we develop the fractional order finite difference schemes for solving these equations.

(vi) To obtain the approximate solutions of fractional partial differential equations with initial and boundary conditions by developed fractional order finite

difference schemes and iterative methods like adomian decomposition method and variational iteration method.

(vii) To represent approximate solutions of fractional partial differential equation graphically by Mathematica software and to discuss their nature.

10 Expected outcome:

We shall develop the fractional finite difference schemes and iterative methods for fractional diffusion equations, fractional wave equations, fractional diffusion-wave equations and similar type of fractional partial differential equations. Also, we will discuss the stability and convergence of the schemes. Furthermore, we obtain the numerical solutions of fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations and similar type of fractional partial differential equations and their solutions shall be represented graphically by Mathematica software.

11 Proposed work plan:

(i) To study partial differential equations and to obtain their analytical solutions.

(ii) To study the diffusion equation and wave equation.

(iii) To develop finite difference schemes and iterative methods for partial differential equations and diffusion-wave equations. Also we obtain the numerical solutions of these equations.

(iv) To study the Mathematica software.

(v) To study the fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations.

(vi) To study the fractional order finite difference schemes and iterative methods.

(vii) To develop fractional order finite difference schemes and iterative methods for fractional diffusion equation, fractional wave equation and fractional diffusion-wave equations. Also we obtain its numerical solutions. Furthermore, we discuss the stability and convergence of these schemes in

detail.

(viii) To study the recent applications of fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations in science and technology.

(ix) To obtain its numerical solutions and simulate these solutions by Mathematica software.

(x) To publish the work in reputed Journals.

12 Year wise plan of work and targets to be achieved:

First Year

(i) To refer the reference books on partial differential equations and numerical methods for solving partial differential equations.

(ii) To study diffusion equation and wave equation.

(iii) To study numerical and analytical methods to solve these equations.

(iv) To visualize solutions by graphs, using the powerful Mathematica software.

Second Year

(i) We will study important definitions and properties of special functions such as Gamma and Beta functions, the complementary error function, Mittag-Leffer function etc. because these functions are directly occurs in the solution of fractional order partial differential equations.

(ii) To study different definitions of fractional derivatives, integrals and fractional partial differential equations.

(iii) To develop the numerical and iterative methods to solve fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations.

Third Year

(i) To discuss the convergence and stability of various developed finite difference schemes.

(ii) To obtain their approximate solutions and simulate by using Mathematica software.

(iii) To study recent applications of fractional diffusion equations, fractional wave equations and fractional diffusion-wave equations in various fields of science and technology.

(iv) We intend to publish our work in reputed journals.

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